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Design of a nonparametric detector for FM signals

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DESIGN OF A NONPARAMETRIC DETECTOR FOR FM SIGNALS

By

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ABSTRACT

The communication engineer has in the past treated the detection problem from a parametric point of view. That is, the signal has been assumed to be of a known deterministic form and the noise of a known statistical form. This approach has proved valuable in many cases, but it is subject to severe limitations, e.g., if either the noise or signal change form, a new detector must be designed. Because of these limitations there is a need to find a more general form of detector.

This study looks at a specific statistic (the Cramér-von Mises statistic) and designs a detector to utilize this statistic. This detector is of the nonparametric class in that a complete knowledge of the signal and noise is not necessary for its operation. This nonparametric detector is then compared to the optimum detector for several specific cases.

The Cramér-von Mises Detector, while requiring more samples than the optimum detector in each case, can efficiently detect signals in an unlimited number of special cases, while the optimum detector can efficiently detect a signal only in the case for which it is designed.

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CHAPTER I

INTRODUCTION

One of the major problems that has always faced the communication engineer is that of detecting a signal buried in a background of noise. The detection problem is concerned with designing a system which will indicate either the presence or absence of a signal in a noisy background. The tracking of a distant target by use of radar or the recovery of coded signals from man made probes in outer space are two examples of such signal detection.

Most of the work done up until this time has been concerned with the design of a system for a signal of known deterministic form and noise of a known statistical form. This type of detection is known as parametric detection. It is subject to many disadvantages, the main ones being that complete knowledge of signal and noise is necessary before the detector can be designed, and then, if either the noise or signal change, the detector will no longer function properly.

Thus, there is need for a wider class of detectors which require much less a priori information than the parametric detectors. This type of detector, when the signal and the noise are not completely known, is called a nonparametric detector (2). The nonparametric detector is based on nonparametric statistical methods which have been well covered in the literature (13-33). These methods have previously been applied to the detection problem (7-9),(34), but only to a very limited extent. These detectors have almost always been limited to detecting only changes in the d-c level of the noise distribution when a signal is added.

In this paper, nonparametric detectors will be considered, which will not only detect special classes of signals (e.g., those which change the d-c level of the noise), but will in general detect any signal which changes the noise distribution in any way (the signals considered are of the additive variety). These nonparametric detectors will then be compared to the optimum parametric detectors, which are optimum for gaussian noise and gaussian signal plus noise. For the purpose of this paper the optimum detector is that detector which takes the fewest number of samples over the sample interval for a given desired accuracy. The method of comparison used here will be the asymptotic relative efficiency, which is a measure of how many more samples one detector needs than the other one for the same accuracy.

After the theory of nonparametric detectors is developed, a nonparametric detector will be used to detect a frequency modulated (FM) signal in the presence of background noise under the condition of low (<1) signal to noise ratio. Since many FM detection problems are restricted to messages which are assumed to be expressed in a binary coded form (Binary Frequency Shift Keyed) or r-ary coded form (Multiple Frequency Shift Keyed), attention here will be limited to these forms.

CHAPTER II

FORMULATION OF THE DETECTION PROBLEM

Given an observed waveform x(t), it is the function of the detector to determine whether x(t) consists of signal plus noise or noise alone. The detector will base its decision on a set of samples of this waveform over some interval. Thus, it is also assumed that the detector is capable of sampling x(t) at $t=t_1, t_2, \ldots, t_N$ giving the set of samples $x_1, x_2, \ldots, x_N$ where $(x=x(t_i), i=1,2,\ldots,N)$. The decision problem can be thought of as a statistical hypothesis testing problem (1). In statistical hypothesis testing there are two different alternatives, represented by the null hypothesis and the alternate hypothesis. The detector can then be thought of as testing the null hypothesis (x(t) is noise alone) against the alternate hypothesis (x(t) is signal plus noise).

All detectors considered here are characterized by their dichotomous decisions, i.e., either the null hypothesis, noise alone, or the alternate hypothesis, signal plus noise, is accepted. The decision is based upon the fact that some function of the sample is greater than or less than some threshold level. This level is predetermined by the number of errors which can be allowed. There are two types of mutually exclusive errors which can be made. They are:

1. the detector decides that a signal is present when there is actually no signal, this will be called a type I error: the probability of such an error will be given the symbol α,
2. the detector decides no signal is present when there is a signal, this is a type II error: the probability of such an error will be given the symbol β.

In the following chapters α will also be called the probability of false alarm, and β will be called the false dismissal probability.

There are many special cases of the detection problem which can be obtained by making various assumptions about the signal and noise statistics. The case most often considered in texts and in literature is that in which both the noise and the signal distributions are assumed to be known exactly. In this case the optimum detector is the Neyman-Pearson or likelihood detector (1,10). The above detector is optimum in the sense that for a given false dismissal probability, β, and for a given false alarm probability, α, the sample size N is a minimum. If the noise is assumed to have a gaussian distribution and the signal is a constant, then the appropriate

Neyman-Pearson detector is the well known t-detector (10). When the variance of the noise is known, the t-detector is given by

$$t = \frac{\left(\sum_{i=1}^{N} \frac{x_i}{N} - \mu_0\right) N^{1/2}}{\sigma_0} \tag{2.1}$$

When the variance of the noise power is not known, the t-detector is given by

$$t = \frac{\left(\sum_{i=1}^{N} \frac{x_i}{N} - \mu_0\right)}{\left\{\sum_{i=1}^{N} \left(x_i - \sum_{i=1}^{N} \frac{x_i}{N}\right)^2 \Big/ \left[N(N-1)\right]\right\}^{1/2}} \tag{2.2}$$

If the variance is known, the t-detector tests the null hypothesis (the waveform x(t) is gaussian with mean μ_0 and variance σ_0^2) x(t) is noise alone, against the alternate hypothesis (the waveform x(t) is gaussian with mean not equal to μ_0 and variance σ_0^2) x(t) is signal plus noise. The null hypothesis is accepted when t has a value below some preset threshold level and rejected when t is above this threshold level. The threshold level is determined in this case by considering t to have a gaussian distribution with mean zero and variance one.
For the case when the variance is unknown, the detector tests the null hypothesis (the waveform

x(t) has mean μ_0 and unknown variance) x(t) is noise alone, against the alternate hypothesis (x(t) has mean not equal to μ_0 and unknown variance) x(t) is signal plus noise. Here again the null hypothesis is accepted if t is below the threshold. However in this case, the threshold is determined by considering t to have a t-distribution with N-1 degrees of freedom.

The only detector discussed so far has been one designed for the detection of a constant signal. If the signal is of the variety which will increase the noise power but leaves the mean alone, then a new detector is necessary. The Neyman-Pearson detector for this case is the chi-square (χ^2)-detector (10). The statistic used for this detector is

$$\frac{\sum_{i=1}^{N}(x_i - \mu_0)^2}{\sigma_0^2} \quad . \tag{2.3}$$

The decision to reject or accept the null hypothesis is made in the same fashion as before, but the threshold in this case is determined by use of the χ^2-distribution with N-1 degrees of freedom. The null hypothesis in this case is represented by x(t) having a gaussian distribution with mean unknown and variance σ_0^2 while the alternate hypothesis is that x(t) has mean unknown and variance greater than σ_0^2.

The Neyman-Pearson detectors described above have proved to be very useful in the past, but as discussed before, there are several major drawbacks to their implementation. First, there must be a good description of both the signal and the noise, and if the noise changes, a new detector must be designed. Second, only one type of signal can be detected with a given detector, and if another signal must be detected, another detector must be designed. Finally, if the noise is not gaussian, it sometimes becomes difficult to implement the Neyman-Pearson detector.

The need for a more general detector leads to a consideration of the so called nonparametric detector. It can be considered to be more general than the parametric detector because a complete description of the signal and noise is not necessary for its utilization. In the following chapters, such a detector will be discussed and compared to the Neyman-Pearson detector.

The criteria used for the comparison will be the asymptotic relative efficiency (A.R.E.) (9,10). Given two detectors, each with the same α and β , the first with sample size N and the second with sample size N*, then the A.R.E. of the second detector with respect to the first is defined by

$$\text{A.R.E.} = \lim_{\theta \to 0} (N/N^*) \tag{2.4}$$

where θ is defined as the signal to noise ratio.

The signal to noise ratio will be defined as the ratio of the r.m.s. value of the signal to the r.m.s. value of the noise. In the following chapters f(x) is defined as the probability density function on x, i.e., the probability of x falling between x and $x+\Delta x$ is $f(x)\Delta x$, and F(x) is defined as the cumulative distribution or the probability that x takes on a value less than or equal to x, i.e.,

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

If f(x) has a gaussian distribution with mean zero and variance one then it will be abbreviated as $f(x) \sim G(0,1)$. If $f(x) \sim G(0,1)$, then the cumulative of f(x) will be denoted by $\hat{G}(x)$. $\hat{G}^{-1}(b)$ will signify the number whose cumulative gaussian distribution, G(0,1), is b.

CHAPTER III

CRAMÉR-VON MISES DETECTOR

3.1 Definition

As has been stated, the detector problem is one of hypothesis testing. The problem now is to test some hypothesis about the observed distribution against that of noise alone. The detector used here is the Cramér-von Mises detector. It is a "goodness of fit" detector. That is, it tests how well the observed distribution matches the assumed distribution. If the fit is good, the null hypothesis (noise alone) is accepted.

It now becomes necessary to find some sort of statistic to use to test the "goodness of fit". In the case of the Cramér-von Mises detector this statistic is the integral over all x of the squared difference between the observed cumulative distribution and the noise alone distribution.

Let the noise have an assumed known distribution f(x) with known cumulative distribution F(x). The detector takes N samples from x(t), the input waveform, the samples being $x_1, x_2, \ldots, x_N$ ($x_i = x(t_i)$, $i=1,2,\ldots,N$). The detector then orders these samples according to magnitude, giving a new sample distribution $X_1, X_2, \ldots, X_N$ where $X_1 \leq X_2 \leq X_3 \leq \ldots \leq X_N$. The observed cumulative distribution $S_N(x)$, is now defined by

$$S_N(x)=\begin{cases}0 & \text{for } x<X_1 \\ j/N & \text{for } X_j \le x < X_{j+1} \qquad j=1,2,\ldots,N-1 \\ 1 & \text{for } X_N \le x\end{cases}.$$

It can now be seen that the observed cumulative distribution function is a step function where each step occurs at the sample points (with each step having a height 1/N). In other words, $NS_N(x)$ is the number of sample points less than or equal to x.

The Cramér-von Mises statistic was first suggested by Cramér and von Mises and later modified by Smirnov. This paper uses the modified statistic defined by Anderson and Darling (13) as

$$\omega^2 = \int_{-\infty}^{\infty} \left[S_N(x)-F(x)\right]^2 dF(x) . \qquad (3.1)$$

If ω^2 is small, the detector will accept the null hypothesis, noise alone, and if it is large, the detector will accept the alternate, signal plus noise, hypothesis.

The statistic given in the above definition is difficult to apply in the actual detection problem. To find a more applicable form of this statistic, look at

$$\omega^2 = \int_{-\infty}^{\infty} \left[S_N(x)-F(x)\right]^2 dF(x)$$

$$\omega^2 = \int_{-\infty}^{X_1} F^2(x)dF(x) + \int_{X_1}^{X_2} \left[S_N(x)-F(x)\right]^2 dF(x) + \cdots + \int_{X_N}^{\infty} \left[1-F(x)\right]^2 dF(x) .$$

Integrating the above, collecting terms, and multiplying through by N, yields

$$N\omega^2 = \sum_{j=1}^{N} \left[F(X_j) - (2j-1)/2N\right]^2 + 1/12N \qquad (3.2)$$

where $F(X_j)$ is the probability that x is less than or equal to X_j . This expression for $N\omega^2$ is the expression actually used by the detector for the decision of signal plus noise or noise alone.

The asymptotic theory for the limiting distribution of $N\omega^2$ is very difficult. However, Smirnov showed that the limiting characteristic function is

$$\phi(t) = \left\{\frac{(2it)^{1/2}}{\sin\left[(2it)^{1/2}\right]}\right\}^{1/2}$$

where

$$\phi(t) = \lim_{N\to\infty} E\left\{\exp(itN\omega^2)\right\}$$

and where $i = \sqrt{-1}$.

The probability density function p(x) for $x = N\omega^2$ can now be found by

$$p(x) = (1/2\pi)\int_{-\infty}^{\infty} \phi(t)\exp(-itx)dt .$$

Thus p(x) is the inverse Fourier transform of $\phi(t)$.

It can be seen from the above that the assumed noise distribution F(x) has no effect upon the distribution for $N\omega^2$. Therefore, the probability of false alarm is independent of F(x), thus $N\omega^2$ is a so called "distribution-free" statistic.

The actual evaluation of p(x), up to now, has been accompolished only by numerical methods. Anderson and Darling (13) give a table of values for the critical point, ω_α, for different values of α, where ω_α and α are defined by $Pr[N\omega^2 \geq \omega_\alpha] = \alpha$ (see Table I). In other words α is the area to the right of ω_α under the curve $p(N\omega^2)$ vs. $N\omega^2$ (see Fig. 3.1). This provides a method for calculating the probability for a type I error, i.e., it provides a method for finding the critical point for a given type I probability of error.

Since a technique for finding the false alarm probability, α, has been found, a technique must be found for finding the false dismissal probability, β. Again look at the definition for $N\omega^2$. Let the null hypothesis have a distribution of F(x), and let the alternate hypothesis have the distribution G(x) and the sampled form of G(x) be denoted by $G_N(x)$. Then if the alternate hypothesis is true, ω^2 looks like

TABLE I

Critical Points for the Cramer-von Mises Statistic

$$1-\alpha = \lim_{N \to \infty} \Pr\left\{ N\omega^2 \leq \omega_\alpha \right\}$$

ω_α	$1-\alpha$
0.02480	0.010
0.03656	0.050
0.04601	0.100
0.05426	0.150
0.06222	0.200
0.07860	0.300
0.09696	0.400
0.11888	0.500
0.14663	0.600
0.18433	0.700
0.24124	0.800
0.34730	0.900
0.36421	0.910
0.38331	0.920
0.40520	0.930
0.43077	0.940
0.46136	0.950
0.49929	0.960
0.54885	0.970
0.61981	0.980
0.74346	0.990
1.16786	0.999

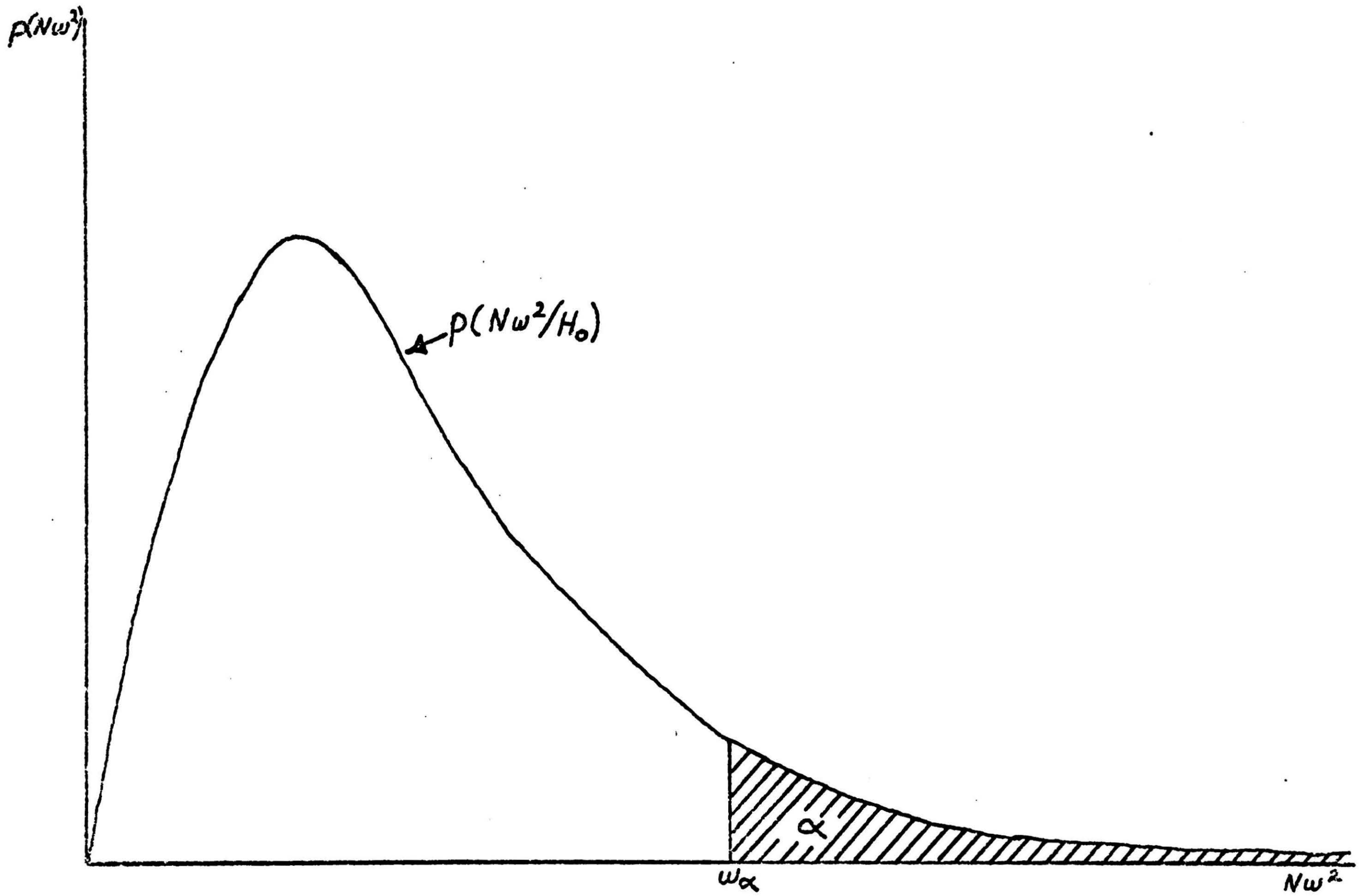

Figure 3.1. Determination of the threshold for the C.V.M. statistic

$$\omega^2 = \int_{-\infty}^{\infty} \left[F(x) - G_N(x)\right]^2 dF(x)$$

$$\omega^2 = \int_{-\infty}^{\infty} \left[F(x) - G(x) + G(x) - G_N(x)\right]^2 dF(x) .$$

Let $\delta(x) = F(x) - G(x)$

then

$$\omega^2 = \int_{-\infty}^{\infty} \delta^2(x) dF(x) + 2 \int_{-\infty}^{\infty} \delta(x) G(x) dF(x) - 2 \int_{-\infty}^{\infty} \delta(x) G_N(x) dF(x) + \int_{-\infty}^{\infty} \left[G(x) - G_N(x)\right]^2 dF(x) . \tag{3.3}$$

If

$$D(x) = \int_{-\infty}^{x} \delta(x) dF(x) \tag{3.3a}$$

and

$$C(G) = \int_{-\infty}^{\infty} \delta^2(x) dF(x) + 2 \int_{-\infty}^{\infty} \delta(x) G(x) dF(x) - 2D(\infty) + 2E\left[D(x)\right] . \tag{3.3b}$$

In Appendix A, it is shown that $\sqrt{N}\left[\omega^2 - C(G)\right]$ is asymptotically normal with mean zero and the following variance

$$\sigma^2(G) = 4 \left\{ E\left[D(x)\right] - \left(E\left[D(x)\right]\right)^2 \right\} .$$

Therefore, $\sqrt{N}\left[\omega^2 - C(G)\right] / \sigma(G)$ has a normal (Gaussian) distribution with mean zero and variance one. Thus, β can now be easily calculated by

$$\beta = \Pr(N\omega^2 < \omega_\alpha) = \int_{-\infty}^{\lambda_1} \phi(x) dx \tag{3.4}$$

where $\phi(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$
and

$$\lambda_1 = \frac{\sqrt{N}\left[\frac{\omega_\alpha}{N} - C(G)\right]}{\sigma(G)} .$$

Since $\sqrt{N}\left[\omega^2 - C(G)\right]/\sigma(G)$ has a normal distribution, G(0,1), and since $\omega^2 \geq 0$ and $\sigma(G)$ is greater than zero, then C(G) must be greater than zero. When N becomes very large, ω_α/N will become less than C(G), thus forcing λ_1 to be a large negative number. As a result as $N \rightarrow \infty$, $\beta \rightarrow 0$.

The expression derived above, Eq. (3.4), for β is dependent upon not only the distance between the cumulative distributions, but also upon the alternate hypothesis itself. It would seem desirable to obtain a more general expression for β. This can be done, but only with the sacrifice of accuracy. Chapman (20) developed an expression for the maximum value of β for one sided hypotheses ($F(x) \geq G(x)$ for all x) with given α and N. This expression is given by

$$\beta_{max} = \int_{-\infty}^{x_1} \phi(x)\,dx$$

where

$$x_1 = \left[\frac{2\omega_\alpha^2}{\Delta^2\sqrt{N}} - \frac{2\Delta}{3}\left(1 + \frac{1}{N}\right)\sqrt{N} - \frac{1}{3\Delta^2\sqrt{N}}\right]$$

and

$$\Delta = \max_{-\infty < x < \infty} \left[F(x) - G(x)\right] .$$

This expression gives a rough estimate for β for large N, but it approaches zero slowly as N increases and is applicable only for a one sided test. So in the following work the original expression Eq. (3.4) for β will be used.

3.2 Asymptotic Relative Efficiency Versus the t-Detector for Detecting d-c Levels

As previously stated, the t-detector is the optimum detector for detecting a constant signal in gaussian noise. This means that for a given probability of false alarm α and false dismissal β the t-detector requires the smallest number of samples.

The asymptotic relative efficiency (A.R.E.) will be used as the goodness criterion for the Cramér--von Mises (C.V.M.) detector. The t-detector is optimum for this case, therefore, its A.R.E. will be taken to be equal to one.

To actually calculate the A.R.E. of the C.V.M. detector, an expression for the number of samples necessary to obtain a given α and β for both the t-detector and the C.V.M. detector must be developed. In each case the noise will be assumed to be gaussian with mean μ_0 and variance σ_0^2. The value of the constant signal will be taken to be A. The signal to noise ratio θ has previously been defined as the ratio of the r.m.s. signal to the r.m.s. noise. The r.m.s. value of a constant is that constant, and the r.m.s. value of a gaussian distribution is σ_0. Thus

$$\theta = A/\sigma_0 \, . \tag{3.5}$$

Each detector tests the null hypothesis (the observed waveform consists of noise alone, gaussian with mean μ_0 and variance σ_0^2) against the alternate hypothesis (the waveform is gaussian with mean not equal to μ_0 and variance σ_0^2). Since A can be either positive or negative, θ can be both positive and negative. Therefore, the tests are called two sided tests.

It is now necessary to find an expression for the number of samples (N*) necessary for the C.V.M. detector to make its decisions with a given α and β. This can easily be done by examining the expression for β Eq.(3.3)

$$\beta = \int_{-\infty}^{\lambda_1} \Phi(x)dx$$

where

$$\lambda_1 = \frac{\sqrt{N^*}\left[{}^{w}\!/\!_{N^*} - C(G)\right]}{\sigma(G)}$$

Using the definitions of Chapter II, the above reduces to

$$\beta = \hat{G}\left\{\frac{\sqrt{N^*}\,[w_\alpha/N^* - C(G)]}{\sigma(G)}\right\}. \tag{3.6}$$

To avoid further confusion due to the introduction of another symbol G, the notation C(G) and σ(G) will be changed. Since for the case under consideration both the null hypothesis and the alternate hypothesis are of fixed form with the only variable being A, both C(G) and σ(G) can be considered functions of A or in general functions of θ. Adopting this new notation Eq.(3.6) becomes

$$\beta = \hat{G}\left\{\frac{\sqrt{N^*}\,[w_\alpha/N^* - C(\theta)]}{\sigma(\theta)}\right\} \tag{3.7}$$

Again using the notation of Chapter II, the following can be obtained

$$\hat{G}^{-1}(\beta) = \frac{\sqrt{N^*}\,[w_\alpha/N^* - C(\theta)]}{\sigma(\theta)}$$

Now solving this expression for N* we obtain

$$N^* = \left\{\frac{-\sigma(\theta)\hat{G}^{-1}(\beta) + \sqrt{[\sigma(\theta)\hat{G}^{-1}(\beta)]^2 + 4w_\alpha C(\theta)}}{2C(\theta)}\right\}^2 \tag{3.8}$$

This gives the number of samples necessary to detect a signal with signal to noise ratio θ and error probabilities α and β.

The actual evaluation of this expression for N* is difficult since the constants $C(\theta)$ and $\sigma(\theta)$ must first be found. This is in general very difficult and even for the relatively simple case of gaussian noise and gaussian signal plus noise, it is necessary to solve for these constants by using numerical methods and a digital computer. See Table II for the results.

It is now necessary to develop an expression for the number of samples necessary for the t-detector. Under the null hypothesis, the observations X_i represent noise alone. Thus, these observations are gaussianly distributed with mean μ_0 and variance σ_0^2

$$X_i \sim G(\mu_0, \sigma_0^2).$$

Since the variance is known, the t-detector is the maximum--likelihood detector for the mean (10), where the estimate of the mean $\hat{\mu}$ is

$$\hat{\mu} = \sum_{i=1}^{N} X / N .$$

For this simple case, it is known that

$$\hat{\mu} \sim G(\mu_0, \sigma_0^2/N)$$

TABLE II

C.V.M. Constants for Use Against the t-Detector

θ	$C(\theta)$	$\sigma(\theta)$
0.6000	3.1479200×10^{-2}	2.8122040×10^{-3}
0.5000	2.2192690×10^{-2}	2.0557732×10^{-3}
0.4000	1.4380200×10^{-2}	1.3721704×10^{-3}
0.3000	8.1673160×10^{-3}	7.9751020×10^{-4}
0.2000	3.6551200×10^{-3}	3.6283130×10^{-4}
0.1000	9.1757100×10^{-4}	9.1989440×10^{-5}
0.0100	9.1856000×10^{-6}	9.2416510×10^{-7}
0.0010	9.1610000×10^{-7}	9.2424480×10^{-9}
0.0001	$8.8400000 \times 10^{-10}$	$9.2402730 \times 10^{-11}$

or

$$\frac{\hat{\mu} - \mu_0}{\sigma_0/\sqrt{N}} \sim G(0,1)$$

The probability of false alarm α is the probability of rejecting the null hypothesis given that the null hypothesis is true. As can be seen in Fig. 3.2, it is possible to write

$$\hat{G}\left[\frac{\hat{\mu}_1 - \mu_0}{\sigma_0/\sqrt{N}}\right] = 1 - \alpha/2$$

or

$$\hat{\mu}_1 - \mu_0 = \frac{\sigma_0\ \hat{G}^{-1}(1-\alpha/2)}{\sqrt{N}} \quad . \qquad (3.9)$$

If the alternate hypothesis is true, the observations represent signal plus noise. The observations are then gaussianly distributed with mean $\mu_0 + A$ and variance σ_0^2, i.e.,

$$X_i \sim G(\mu_0 + A,\ \sigma_0^2) \ .$$

Similiarly

$$\frac{\hat{\mu} - (\mu_0 + A)}{\sigma_0/\sqrt{N}} \sim G(0,1) \ .$$

The probability of false dismissal is the probability of accepting the null hypothesis given that the alternate is true, Therefore, it is possible to write

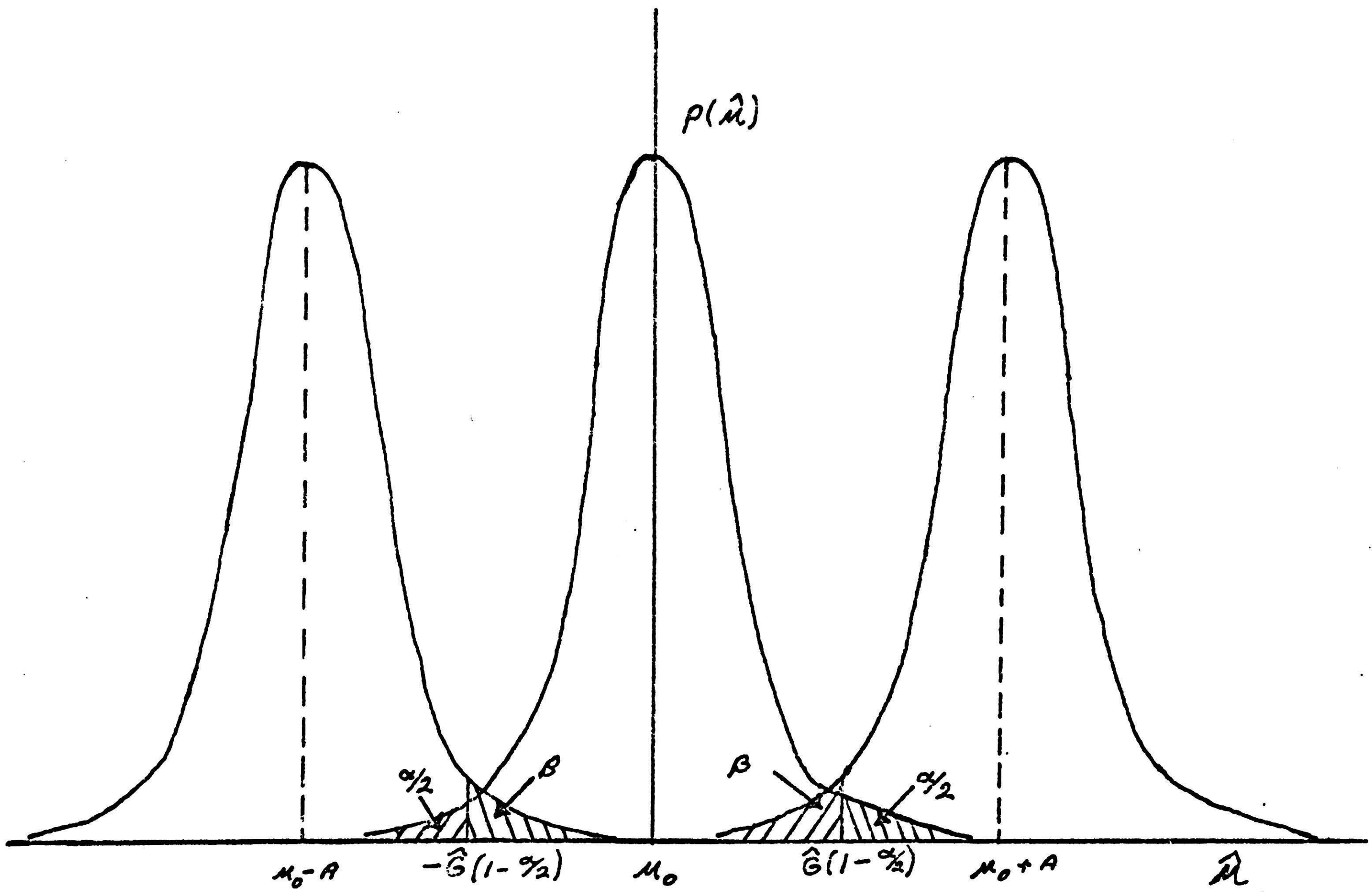

Figure 3.2. Determination of α and β for a Two Sided t-Detector

$$\beta = \hat{G}\left[\frac{\hat{\mu}_1 - (\mu_0 + A)}{\sigma_0/\sqrt{N}}\right]$$

or as before

$$\hat{\mu}_1 - \mu_0 = A + \frac{\sigma_0 \hat{G}^{-1}(\beta)}{\sqrt{N}} . \qquad (3.10)$$

The previous statement is also represented in Fig. 3.2.

The expression for N as a function of α, β, and θ can now be found by setting the right hand sides of Eq.(3.10) and Eq.(3.9) equal to each other, yielding

$$\frac{\sigma_0 \hat{G}^{-1}(1-\alpha/2)}{\sqrt{N}} = A + \frac{\sigma_0 \hat{G}^{-1}(\beta)}{\sqrt{N}} .$$

Substituting the value of A in Eq.(3.5) and solving for N yields

$$N = \left[\frac{\hat{G}^{-1}(1-\alpha/2) - \hat{G}^{-1}(\beta)}{\theta}\right]^2 . \qquad (3.11)$$

The desired expression for the A.R.E. can now be obtained. Using the values of N and N* from Eq.(3.11) and Eq.(3.8) respectively in the definition for the A.R.E., the following is obtained

$$\text{A.R.E.} = \lim_{\theta \to 0}(N/N^*)$$

$$\text{A.R.E.} = \lim_{\theta \to 0}\left\{\frac{2c(\theta)\left[\hat{G}^{-1}(1-\alpha/2) - \hat{G}^{-1}(\beta)\right]}{\theta\left[-\sigma(\theta)\hat{G}^{-1}(\beta) + \sqrt{[\sigma(\theta)\hat{G}^{-1}(\beta)]^2 + 4w_\alpha c(\theta)}\right]}\right\}^2 . \qquad (3.12)$$

This result is plotted as a function of θ for given values of α and β (See Figs. 3.3-3.6). These graphs seem to approach a limit for small θ. Therefore, the A.R.E. corresponding to the smallest value of θ calculated will be considered to be the true A.R.E.. This method may be in error, but the error is felt to be small (Refer again to Figs. 3.3-3.6) and the values picked are less than or equal to the true A.R.E.. Table III gives a final picture of the A.R.E. as a function of α and β, and the A.R.E. seems to approach .550 as α and β go to zero.

It seems reasonable that if A were guaranteed to always be positive, the t-detector could be made more efficient. In the previous case, the t-detector had to detect both positive and negative means. This seems to suggest that some positive information would be lost in using this two sided detector. In the following section, an expression for the A.R.E. of this one sided detector will be developed.

The expression for N* will not be recalculated since it would be extremely difficult to evaluate the density function for $N\omega^2$ under the assumption of a one sided test. An expression for N must now be found. The only difference between this case and the one previously developed is that in

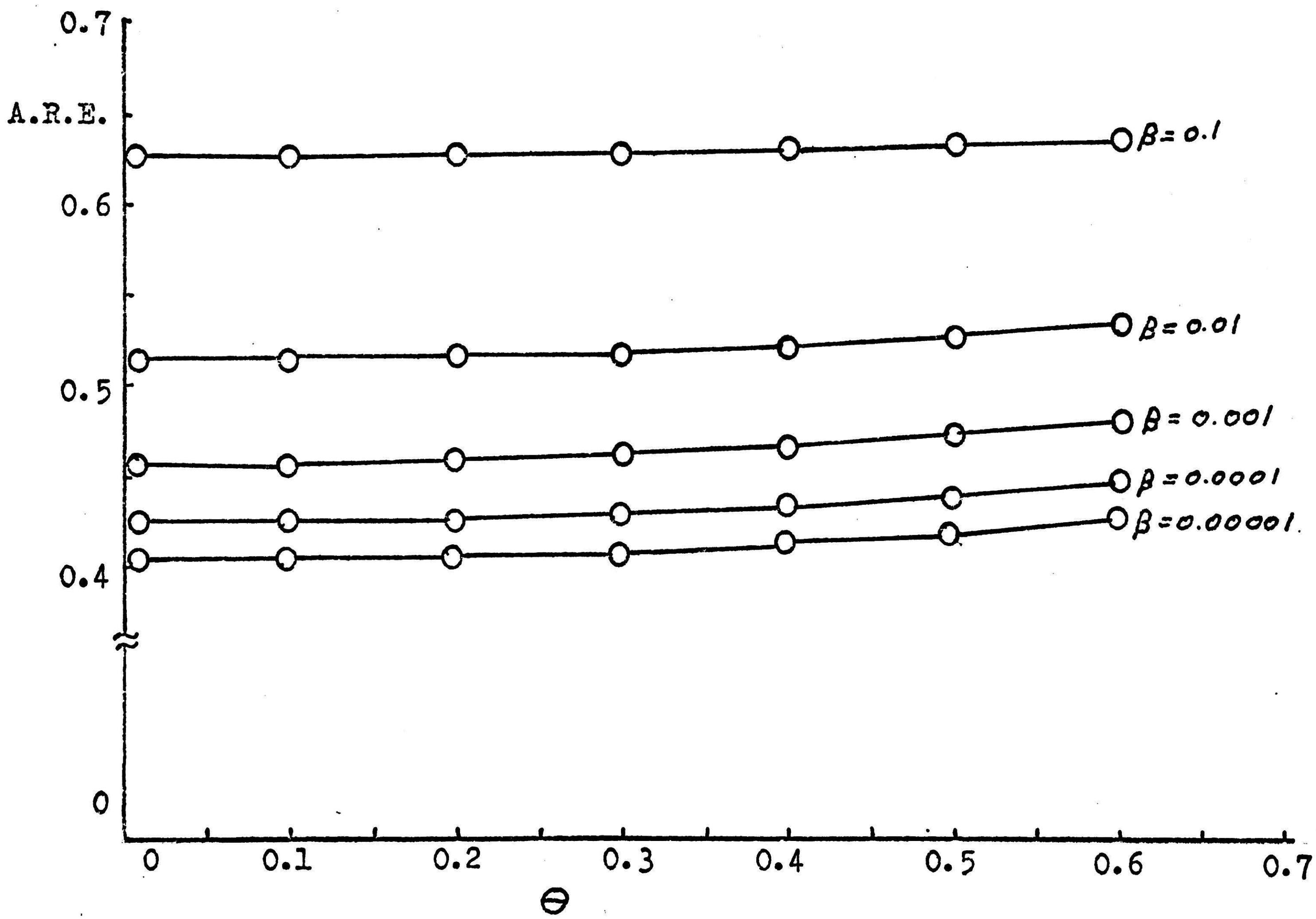

Figure 3.3. A.R.E. vs θ for Two Sided t-Detector; $\alpha = 0.1$

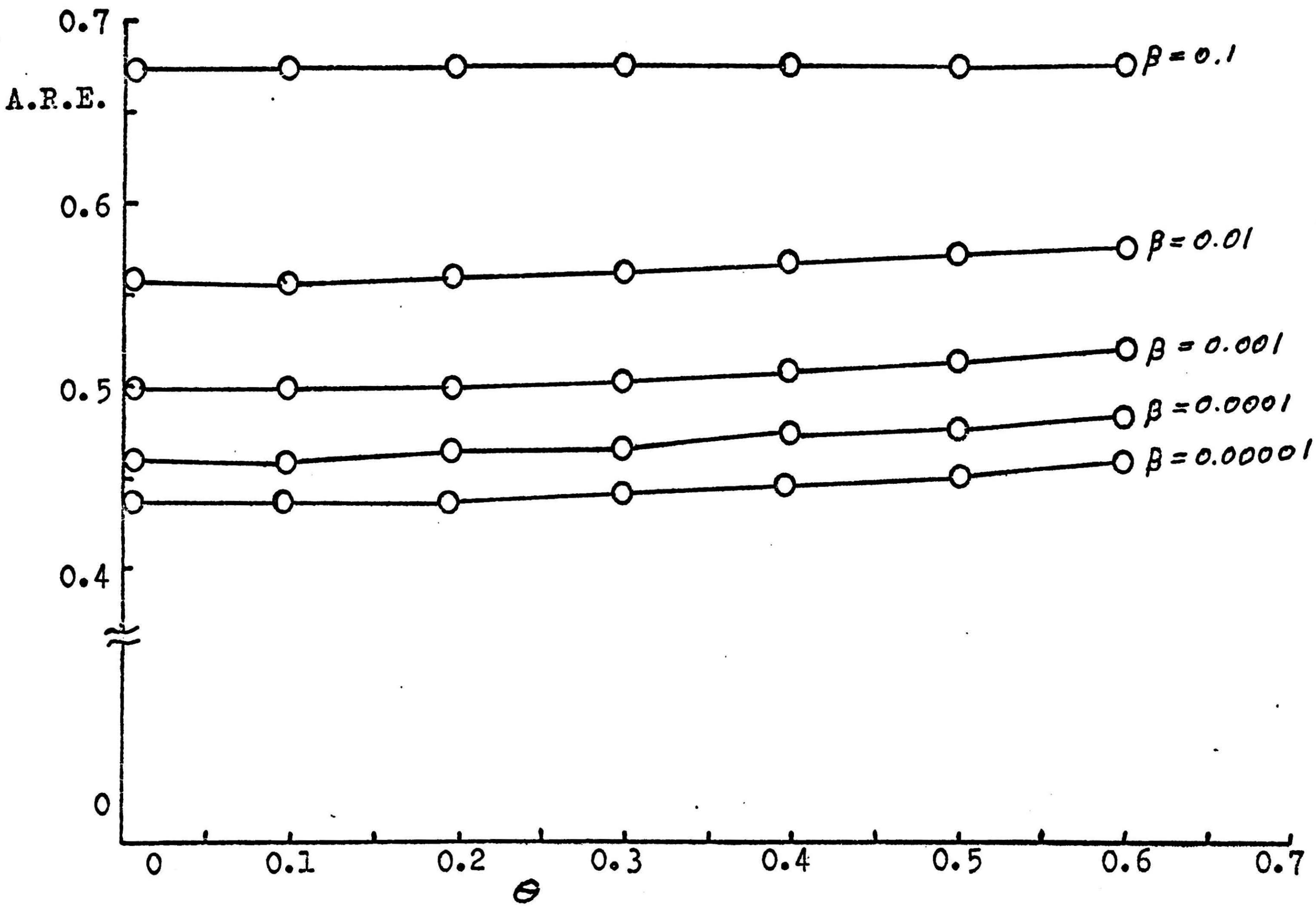

Figure 3.4. A.R.E. vs θ for Two Sided t-Detector; $\alpha = 0.05$

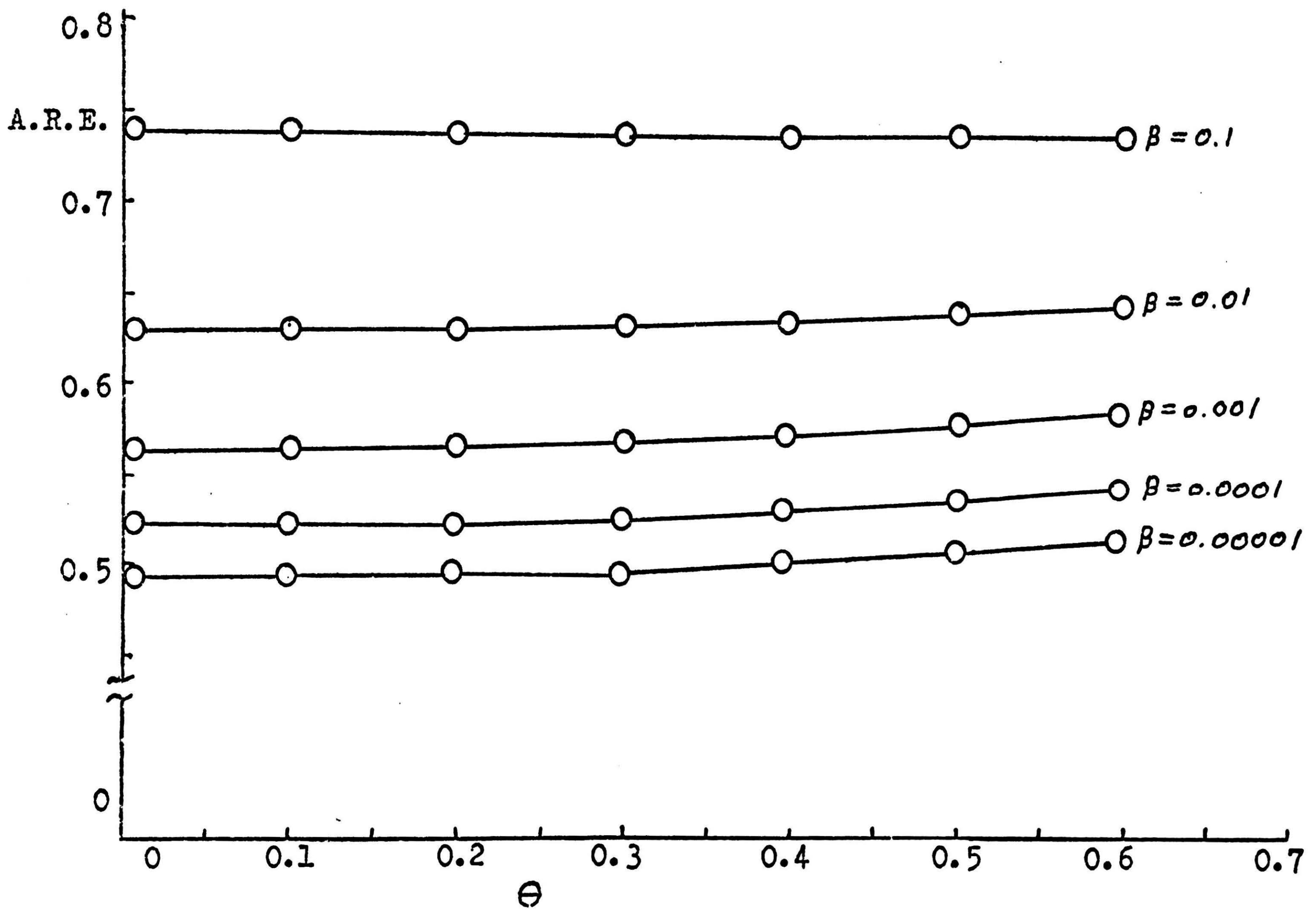

Figure 3.5. A.R.E. vs θ for Two Sided t-Detector; α =0.01

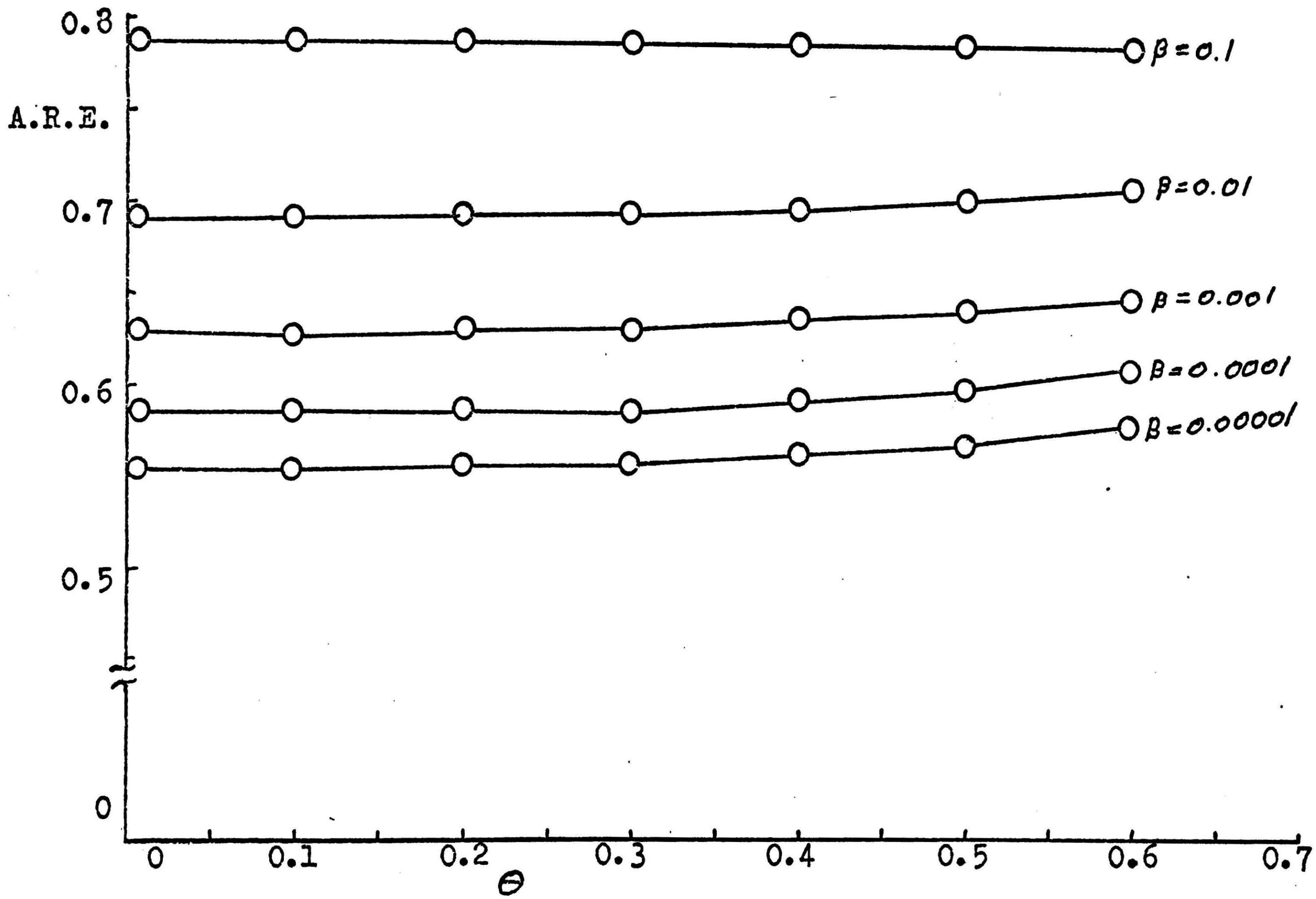

Figure 3.6. A.R.E. vs θ for Two Sided t-Detector; α =0.001

TABLE III

A.R.E. Against the t-Detector

β \ α	0.100	0.050	0.010	0.001
0.10000	0.625	0.671	0.740	0.988
0.01000	0.513	0.556	0.629	0.691
0.00100	0.455	0.498	0.564	0.629
0.00010	0.422	0.459	0.521	0.584
0.00001	0.402	0.435	0.491	0.555

the present case α is under only one tale of the null distribution (See Fig. 3.7). The expression corresponding to Eq.(3.9) would then be

$$\hat{\mu}_1 - \mu_0 = \frac{\sigma_0 \hat{G}^{-1}(1-\alpha)}{\sqrt{N}} \quad . \tag{3.13}$$

Setting Eq.(3.13) equal to Eq.(3.10) yields

$$\hat{\mu}_1 - \mu_0 = \frac{\sigma_0 \hat{G}^{-1}(1-\alpha)}{\sqrt{N}} = A + \frac{\sigma_0 \hat{G}^{-1}(\beta)}{\sqrt{N}}$$

or

$$N = \left[\frac{\hat{G}^{-1}(1-\alpha) - \hat{G}^{-1}(\beta)}{\theta}\right] \quad . \tag{3.14}$$

Now looking at the definition for A.R.E., Eq.(2.5)

$$\text{A.R.E.} = \lim_{\theta \to 0}(N/N^*)$$

it is possible to substitute in Eq.(3.14) and Eq.(3.8) to obtain

$$\text{A.R.E.} = \lim_{\theta \to 0} \left\{ \frac{2c(\theta)\left[\hat{G}^{-1}(1-\alpha) - \hat{G}^{-1}(\beta)\right]}{\theta\left[-\sigma(\theta)\hat{G}^{-1}(\beta) + \sqrt{[\sigma(\theta)\hat{G}^{-1}(\beta)]^2 + 4\omega_\alpha c(\theta)}\right]} \right\}^2 \quad . \tag{3.15}$$

The A.R.E. for this detector is only slightly smaller than the one for the two sided detector.

3.3 Asymptotic Relative Efficiency Against the Chi-Squared Detector for Detecting Power Level Changes

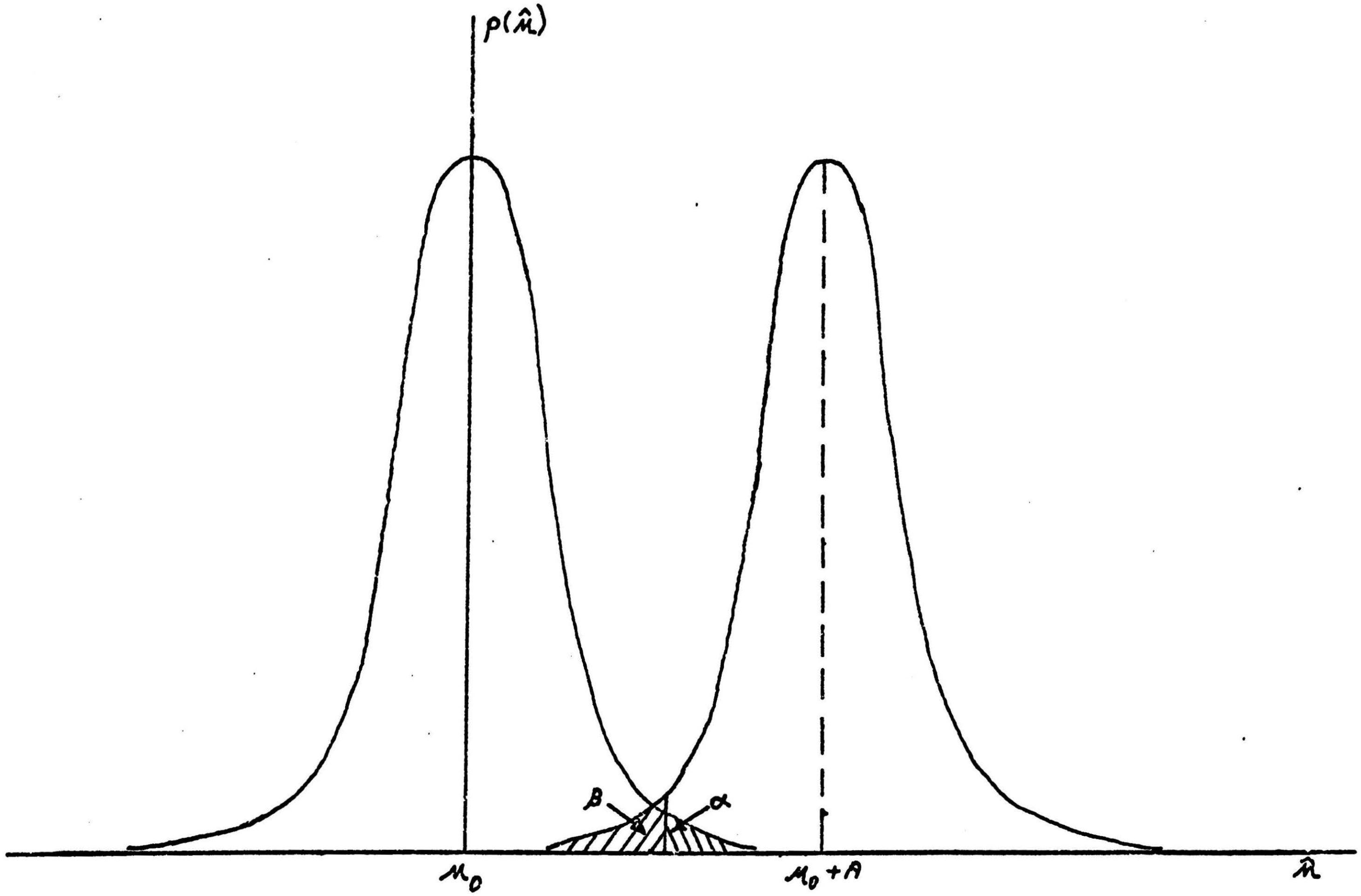

Figure 3.7. Determination of α and β for One Sided t-Detector

The χ^2-detector is the optimum detector for detecting a signal which changes the variance of the data for gaussian noise. It is again optimum in the sense that it requires the fewest number of samples to achieve a given set of error probabilities α and β. The measure used in the comparison of the χ^2-detector and the Cramér-von Mises detector will again be the A.R.E.

The procedure for calculating the A.R.E. will be the same as in the previous section. That is, relationships for N and N* will be developed, and then the ratio N/N* will be examined as θ approaches zero. For the situation where the variance is increased by the addition of the signal, θ is given by

$$\theta^2 = \frac{\sigma_s^2}{\sigma_0^2} = \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2} \qquad (3.16)$$

where σ_s^2 and σ_0^2 are the variances of the signal and noise respectively, while σ_1^2 is the variance of the signal plus noise. Here, each detector can be considered as a test of the null hypothesis, that the observed waveform is noise alone (gaussian with mean μ_0 and variance σ_0^2) against the alternate hypothesis, that the observed waveform is signal plus noise (gaussian with mean μ_0 and variance σ_1^2). Since the addition of any density function to a given density function either increases the variance or leaves it unchanged, the variance of the signal plus noise σ_1^2 will always be greater than or equal to the variance of the noise alone σ_0^2. Thus the signal to noise ratio θ will always be positive and the test will be one sided.

As for the previous case, the number of samples for the χ^2-detector will be denoted by N, and the number of samples for the C.V.M. detector will be denoted by N*. The previously developed relationship for N* (See Eq.(3.8)), namely,

$$N^* = \left[\frac{-\sigma(\theta)\,\hat{G}^{-1}(\beta) + \sqrt{[\sigma(\theta)\hat{G}^{-1}(\beta)]^2 + 4\omega_\alpha C(\theta)}}{2C(\theta)}\right]^2$$

is still valid even though the detector must in this problem detect a change in variance. For this case $\sigma(\theta)$ and $C(\theta)$ are calculated as in the previous section, but here the two densities f(x) and g(x) differ in variance rather than in mean. Again as in the previous section, a digital computer was used to calculate $\sigma(\theta)$ and $C(\theta)$ for several values of θ (See Table IV).

Now an expression for N will be developed. The observations are independent and gaussianly distributed with mean μ_0 and variance $(\theta^2 + 1)\sigma_0^2$.

TABLE IV

C.V.M. Constants for Use Against the χ^2-Detector

θ	$C(\theta)$	$\sigma(\theta)$
0.60	7.1921945×10^{-4}	1.8246283×10^{-5}
0.50	3.7992001×10^{-4}	9.4686111×10^{-6}
0.40	1.6839057×10^{-4}	4.1243102×10^{-6}
0.30	5.6829304×10^{-5}	1.3701219×10^{-6}
0.20	1.1773314×10^{-5}	2.8036374×10^{-7}
0.10	7.5669959×10^{-7}	1.7896596×10^{-8}
0.01	8.0945028×10^{-11}	1.7833799×10^{-12}

Under the null hypothesis θ equals zero, thus the variance is σ_0^2. The distribution of the statistic

$$\frac{\sum_{i=1}^{N}(X_i-\mu_0)^2}{[\theta^2+1]\,\sigma_0^2}$$

is then a chi-squared distribution with N degrees of freedom, i.e.,

$$\frac{\sum_{i=1}^{N}(X_i-\mu_0)^2}{[\theta^2+1]\,\sigma_0^2} \sim \chi^2(N). \tag{3.17}$$

The mean of this chi-squared distribution is N and the variance is 2N. By using the central limit theorem, it can be shown that for large values of N, the chi-squared distribution approaches a gaussian distribution with mean N and variance 2N. Thus

$$\frac{\sum_{i=1}^{N}(X_i-\mu_0)^2}{[\theta^2+1]\,\sigma_0^2} \sim G(N, 2N)$$

for large N.

The statistic used in the χ^2-detector is the maximum likelihood estimator for the variance, $\hat{\sigma}^2$, i.e.,

$$\hat{\sigma}^2 = \sum_{i=1}^{N}(X_i-\mu_0)^2/N \tag{3.18}$$

$\hat{\sigma}^2$ can be shown to have a gaussian distribution, i.e.,

$$\hat{\sigma}^2 \sim G\left([\theta^2+1]\,\sigma_0^2,\ 2[\theta^2+1]^2\sigma_0^4/N\right)$$

thus

$$\frac{\hat{\sigma}^2 - (\theta^2+1)\sigma_0^2}{(\theta^2+1)\sigma_0^2(2/N)^{1/2}} \sim G(0,1).$$

Since α is the probability of accepting the alternate hypothesis, signal plus noise, given that the null hypothesis, noise alone, is true, it is possible to write (See Fig. 3.8)

$$\hat{G}\left[\frac{\hat{\sigma}_1^2 - \sigma_0^2}{\sigma_0^2(2/N)^{1/2}}\right] = 1 - \alpha. \tag{3.19}$$

Similarly, β, the probability of accepting the null hypothesis given that the alternate hypothesis is true can be written as (See Fig. 3.8)

$$\hat{G}\left[\frac{\hat{\sigma}_1^2 - (\theta^2+1)\sigma_0^2}{(\theta^2+1)\sigma_0^2(2/N)^{1/2}}\right] = \beta. \tag{3.20}$$

Solving for $\hat{\sigma}_1^2$, in both Eq.(3.19) and Eq.(3.20), and setting the results equal gives

$$\hat{\sigma}_1^2 = \sigma_0^2 + \sigma_0^2(2/N)^{1/2}\hat{G}^{-1}(1-\alpha) = (\theta^2+1)\sigma_0^2 + (\theta^2+1)\sigma_0^2(2/N)^{1/2}\hat{G}^{-1}(\beta)$$

Then solving for θ^2 yields

$$\theta^2 = [\hat{G}^{-1}(1-\alpha) - \hat{G}^{-1}(\beta)]/[(N/2)^{1/2} + \hat{G}^{-1}(\beta)].$$

For large N, $\hat{G}^{-1}(\beta)$ can be neglected with respect to $(N/2)^{1/2}$. Now rearranging the above and squaring

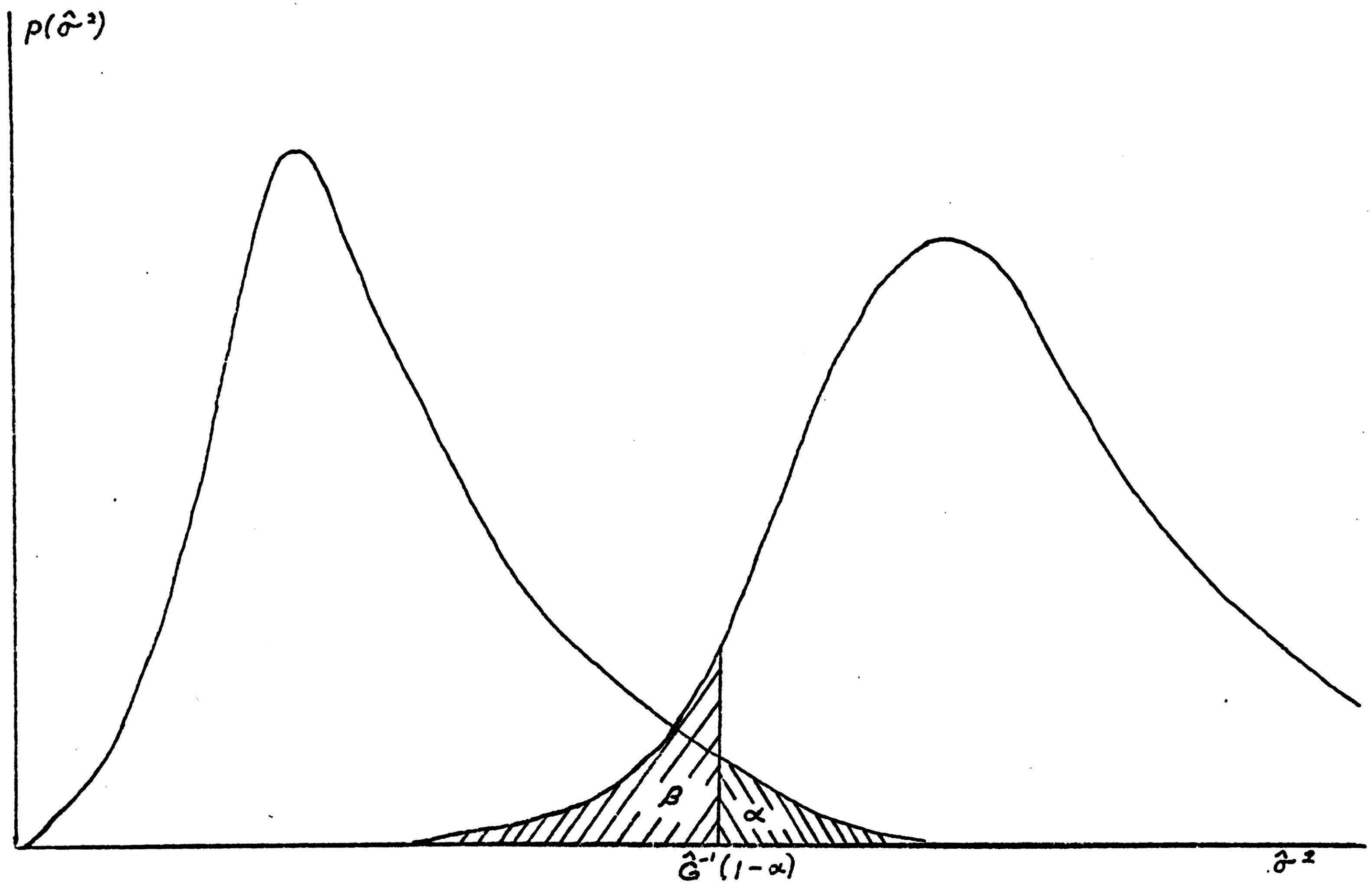

Figure 3.8. Determination of α and β for χ^2-Detector

yields

$$N\theta^4 = 2\left[\hat{G}^{-1}(1-\alpha) - \hat{G}^{-1}(\beta)\right]^2$$

or

$$N = \frac{2\left[\hat{G}^{-1}(1-\alpha) - \hat{G}^{-1}(\beta)\right]^2}{\theta^4} \quad . \tag{3.21}$$

Eq.(3.21) gives the desired relationship for N, the number of samples necessary for the χ^2-detector.

The A.R.E. can now be calculated by substituting the relationships for N*, Eq.(3.8), and N, Eq.(3.21), into the definition of the A.R.E., Eq.(2.4). Carrying out the above steps yields

$$\text{A.R.E.} = \lim_{\theta \to 0} 2\left\{\frac{2c(\theta)\left[\hat{G}^{-1}(1-\alpha)-\hat{G}^{-1}(\beta)\right]}{\theta^2\left[-\sigma(\theta)\hat{G}^{-1}(\beta)+\sqrt{[\sigma(\theta)\hat{G}^{-1}(\beta)]^2+4\,w_\alpha c(\theta)}\right]}\right\}^2 . \tag{3.22}$$

As in the previous case, the value of the A.R.E. was investigated as θ approached zero, for given values of α and β (See Figs. 3.9-3.12). For each value of α and β as θ gets smaller the A.R.E. increases. For a tabulation of these values see Table V. In each case the A.R.E. is taken to be the value corresponding to the smallest calculated θ for a given α and β. As α and β approach zero the A.R.E. of the C.V.M. detector is at least 0.245.

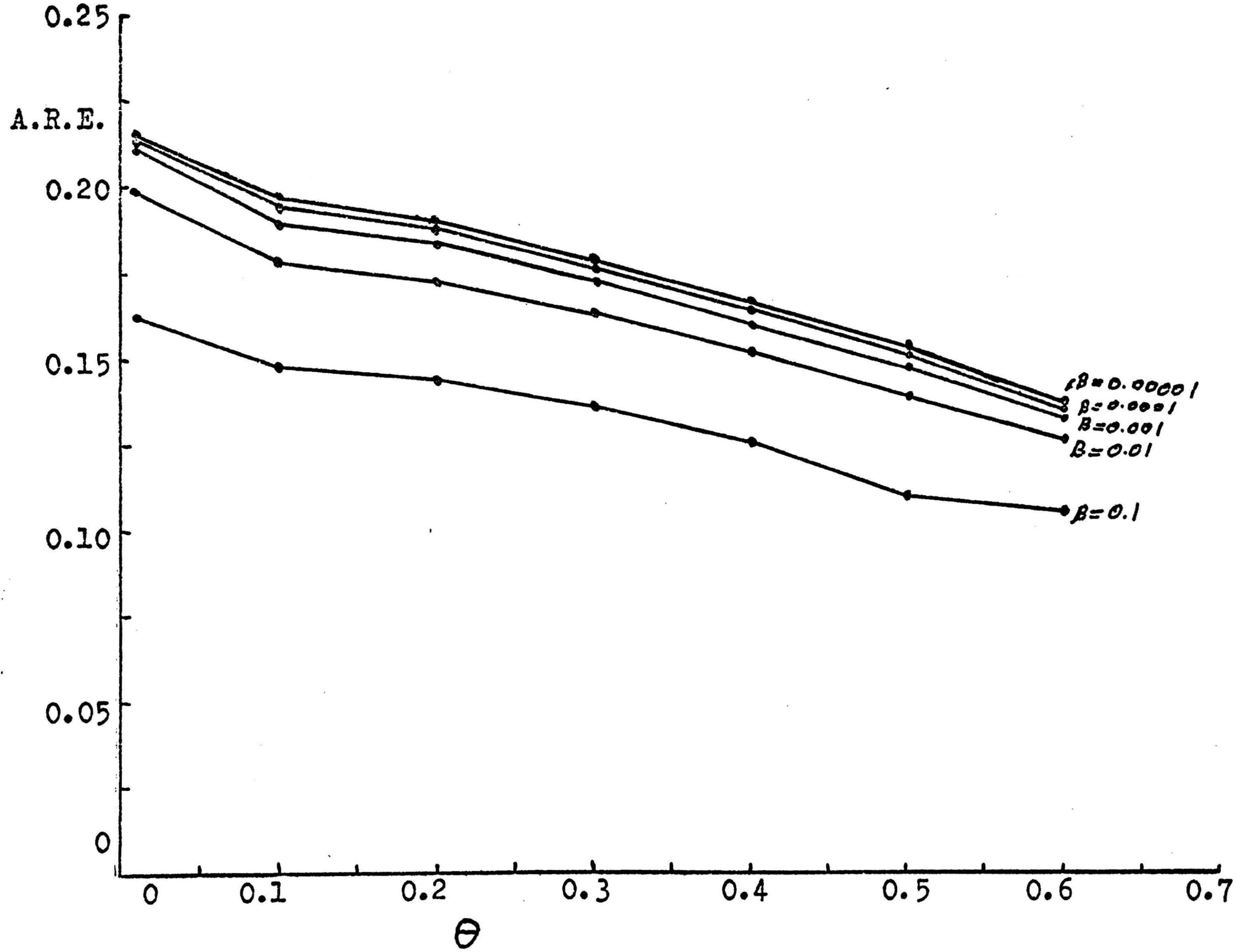

Figure 3.9. A.R.E. vs θ for χ^2-Detector; α=0.1

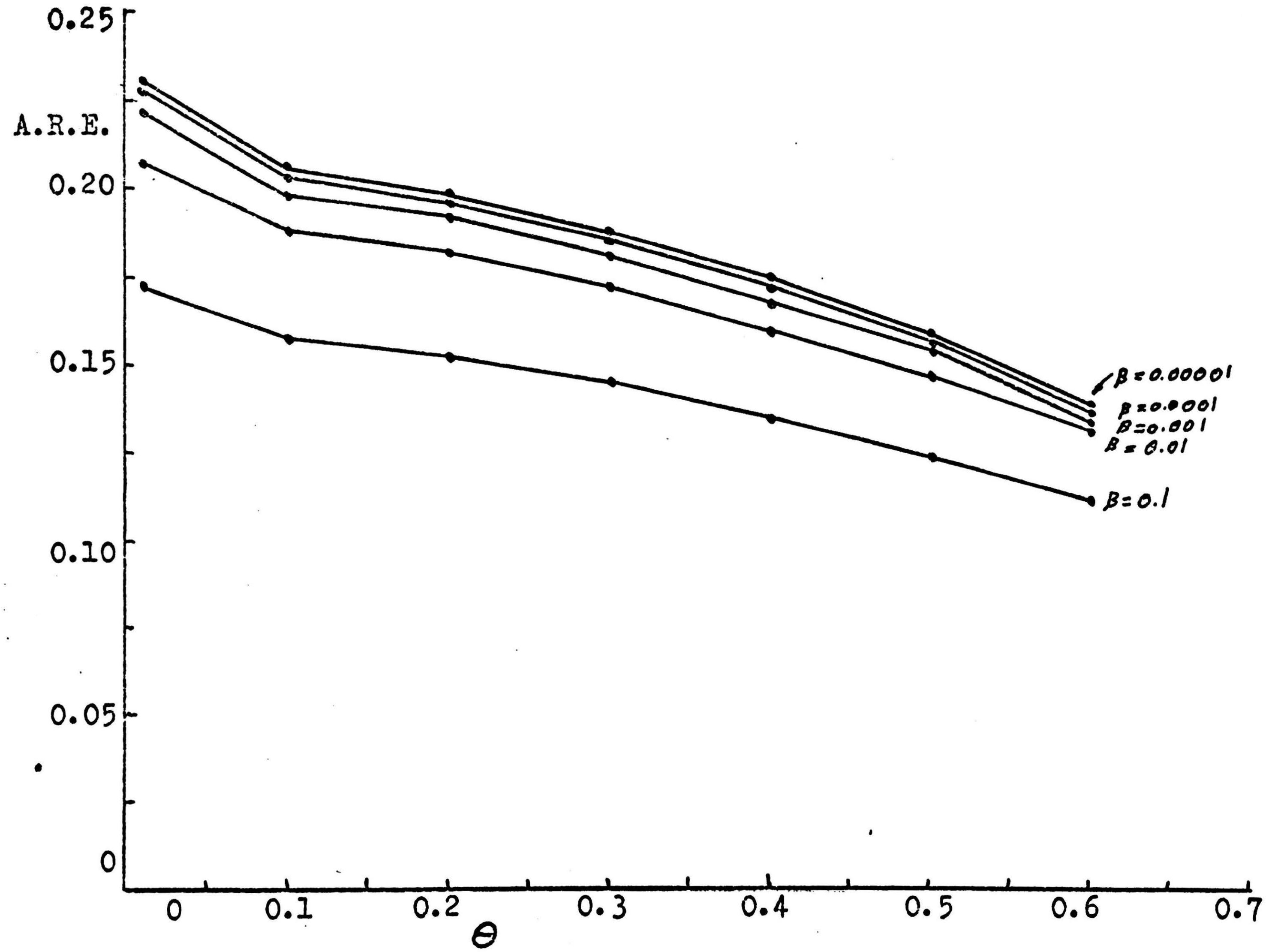

Figure 3.10. A.R.E. vs θ for χ^2-Detector; α =0.05

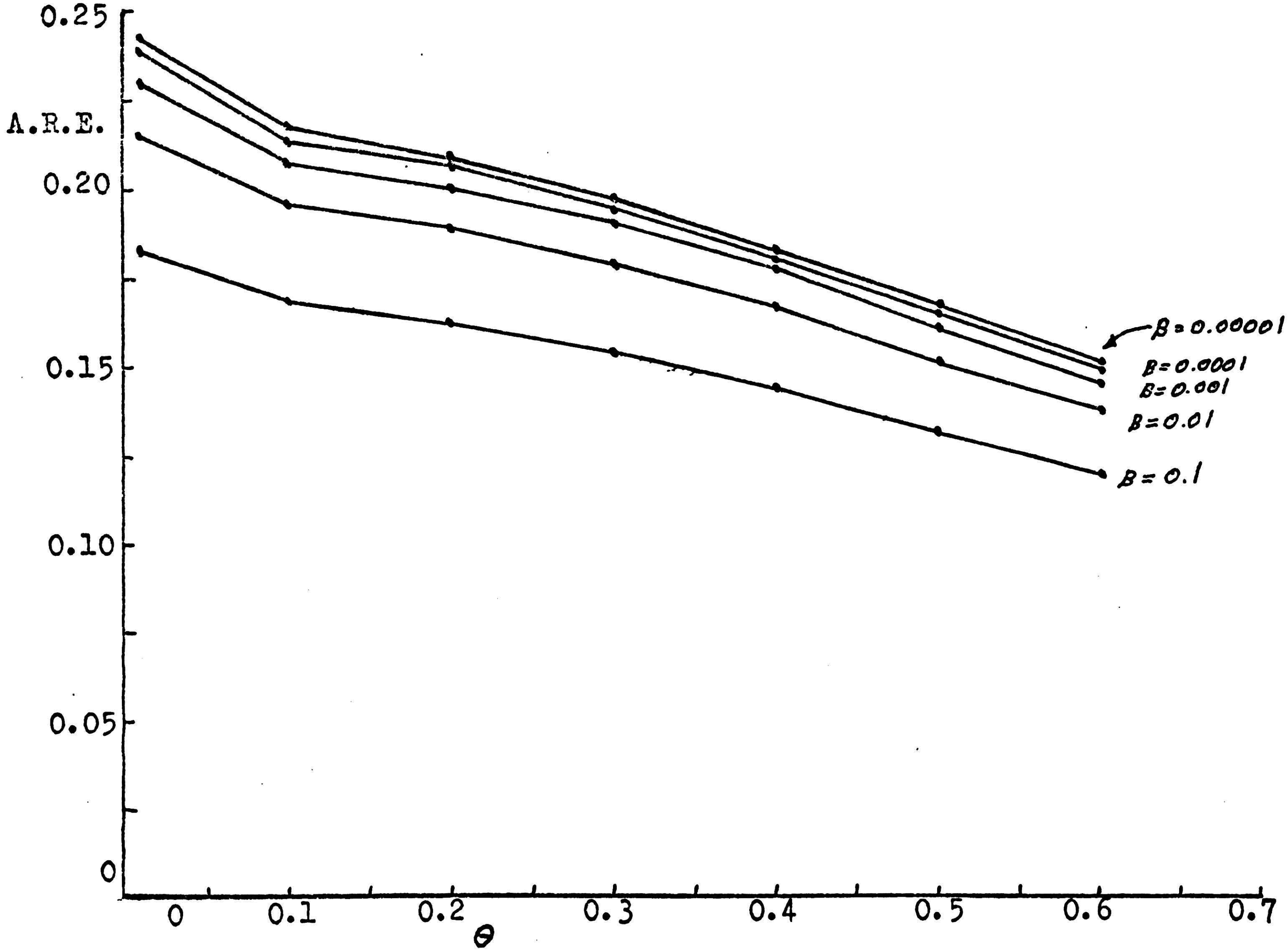

Figure 3.11. A.R.E. vs θ for χ^2-Detector; α =0.01

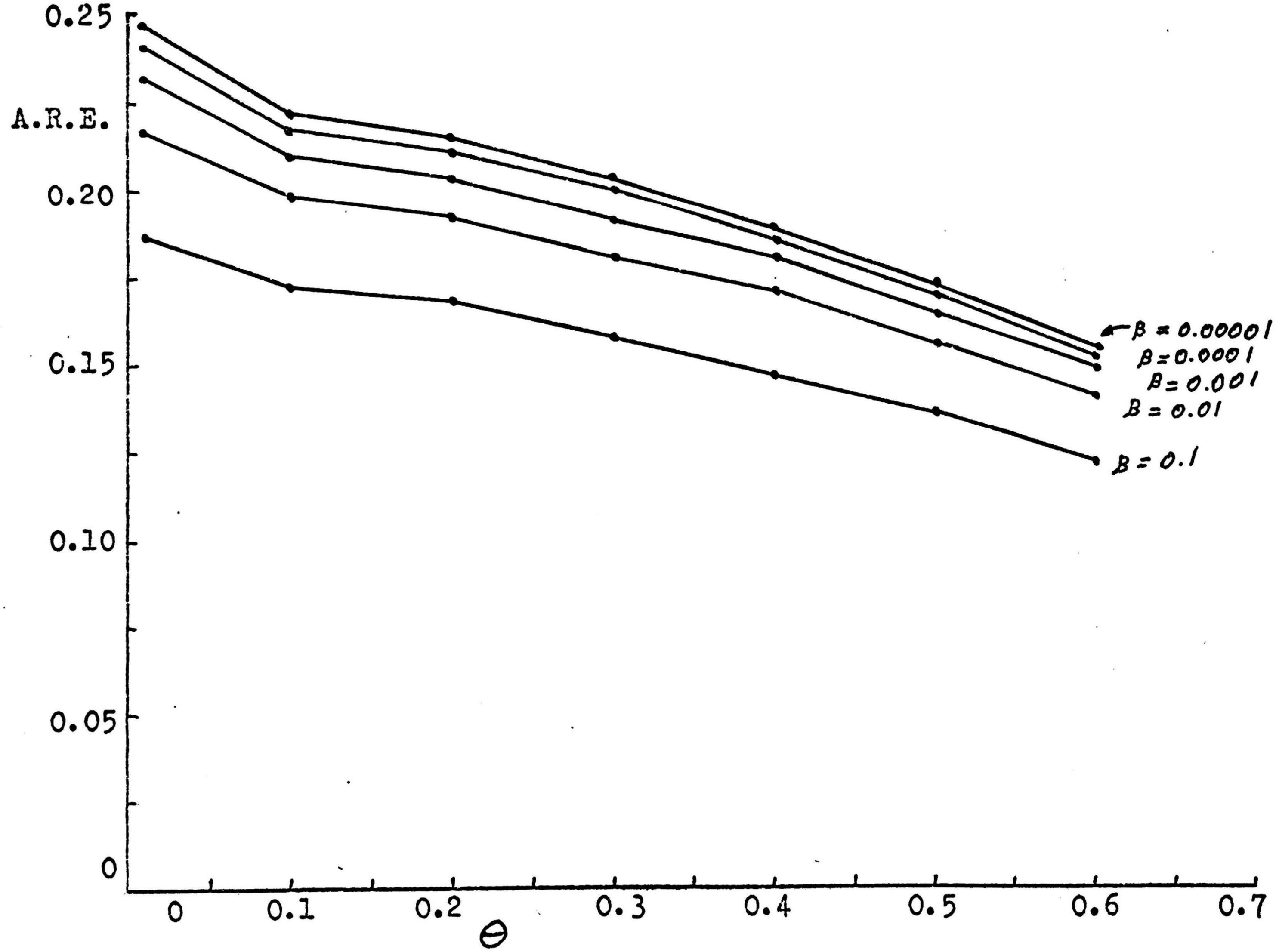

Figure 3.12. A.R.E. vs θ for -Detector; $\alpha = 0.001$

TABLE V

A.R.E. Against the χ^2-Detector

β \ α	0.100	0.050	0.010	0.001
0.10000	0.162	0.172	0.181	0.186
0.01000	0.198	0.208	0.215	0.217
0.00100	0.211	0.222	0.230	0.232
0.00010	0.214	0.228	0.238	0.240
0.00001	0.215	0.231	0.242	0.246

CHAPTER IV

DETECTION OF AN FM SIGNAL

Previously, the problem of detecting a signal has only been discussed in general terms. In this chapter, a C.V.M. detector will be used to detect a given FM signal. The noise will be assumed to be gaussian. If the noise distribution is not known, it can be found by sampling the noise alone over a sufficient length of time to give the noise distribution to any desired accuracy. The problem here is for the detector to decide with a given false alarm and false dismissal probability, whether the FM signal is present or noise alone is present.

It is desired to detect the presence or absence of a general message m(t). This message frequency modulates a cosine wave carrier, to give

$$s(t) = A\cos\left[\omega_c t + \int_0^t m(z)dz\right] \tag{4.1}$$

where A is the amplitude and ω_c is the carrier frequency. The above, Eq.(4.1), is the basic form for a frequency modulated wave. To add more generality, it is necessary to consider both amplitude fading and phase fading. In so doing Eq.(4.1) is replaced by

$$s(t) = a(t)\cos\left[\omega_c t + \int_0^t m(z)dz + \phi(t)\right] \tag{4.2}$$

where a(t) is the amplitude fading term and $\phi(t)$ is the phase fading term.

In Chapter I it was stated that for many important problems the messages can be restricted to those expressed in a binary coded form (Binary Frequency Shift Keying) or in a r-ary coded form (Multiple Frequency Shift Keying). Therefore, attention here will be restricted to these forms. In the Binary Frequency Shift Keying (B.F.S.K.) the message is of the form

$$m(z) = 0$$

or

$$m(z) = \Delta\omega_c t$$

and in Multiple Frequency Shift Keying (M.F.S.K.) the message is of the form

$$m(z) = 0$$

or

$$m(z) = \Delta_1\omega_c t$$

or

.

.

or

$$m(z) = \Delta_r \omega_c t$$

If the message is limited to the cases above, i.e., B.F.S.K. or M.F.S.K., the modulated carrier s(t) can be rewritten as

$$s(t) = a(t)\cos\left[\omega_j t + \phi(t)\right] \qquad (4.3)$$

where

$$\omega_j = \omega_c + \Delta_j \omega_c \qquad j=0,1,2,\ldots,r$$

and where Δ_0 equals zero. To detect a message of the form given in Eq.(4.3) a filter must be centered at each frequency ω_j and followed by a C.V.M. detector to detect the presence or absence of a signal at each ω_j.

The problem now reduces to the sampling of the waveform x(t) and deciding if

$$x(t) = n(t)$$

noise alone or if

$$x(t) = a(t)\cos\left[\omega_j t + \phi(t)\right] + n(t)$$

signal plus noise. The functions a(t) and $\phi(t)$ can be assumed to be slow varying compared to $\omega_j t$, therefore they are assumed to be time stationary.

The noise is also assumed to be time stationary, but the waveform x(t) can not be assumed stationary because of $\omega_j t$.

The signal detected in this chapter will be assumed to be subject to Rayleign amplitude fading and uniform phase fading. The signal to noise ratio for this case is shown in Appendix B to be

$$\theta = \frac{\sigma_s}{\sigma_0}$$

where σ_s is the parameter in the Rayleigh Distribution.

A technique must now be found to obtain the number of samples necessary for the C.V.M. detector to operate with a given β. Since in the previous work no expression has been found for a β_{max} independent of the signal plus noise distribution function, an expression for this distribution function must be found. Using the assumptions stated above regarding the signal, assuming the noise has a gaussian distribution, and assuming a signal to noise ratio of one-half the probability density function of the signal plus noise was calculated using a digital computer (See Fig. 4.1). The number of samples necessary for the C.V.M. detector was then calculated using the expression for N* developed previously, Eq. (3.8), (See Fig. 4.2).

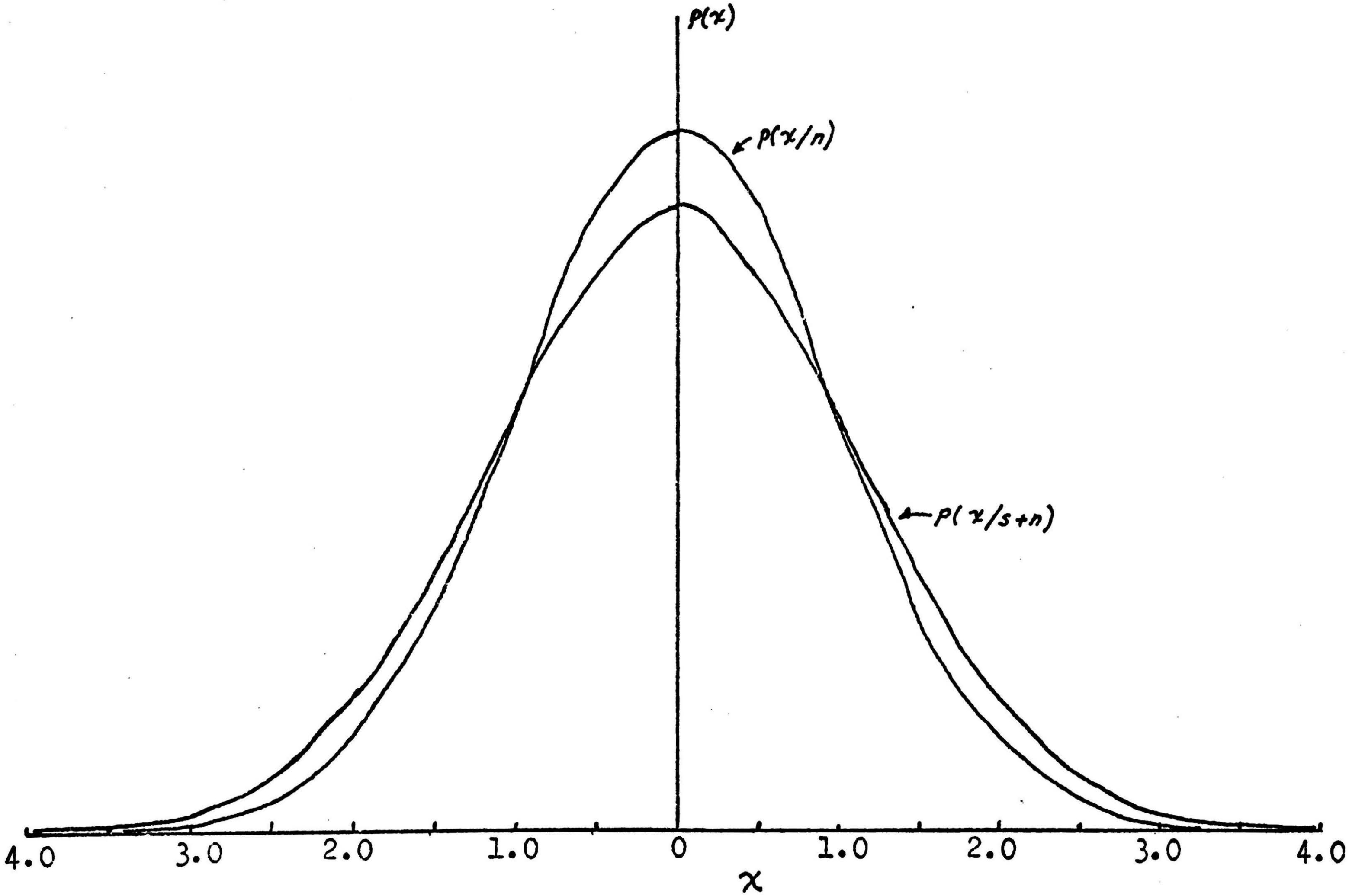

Figure 4.1. Probability Density of FM Signal Plus Noise

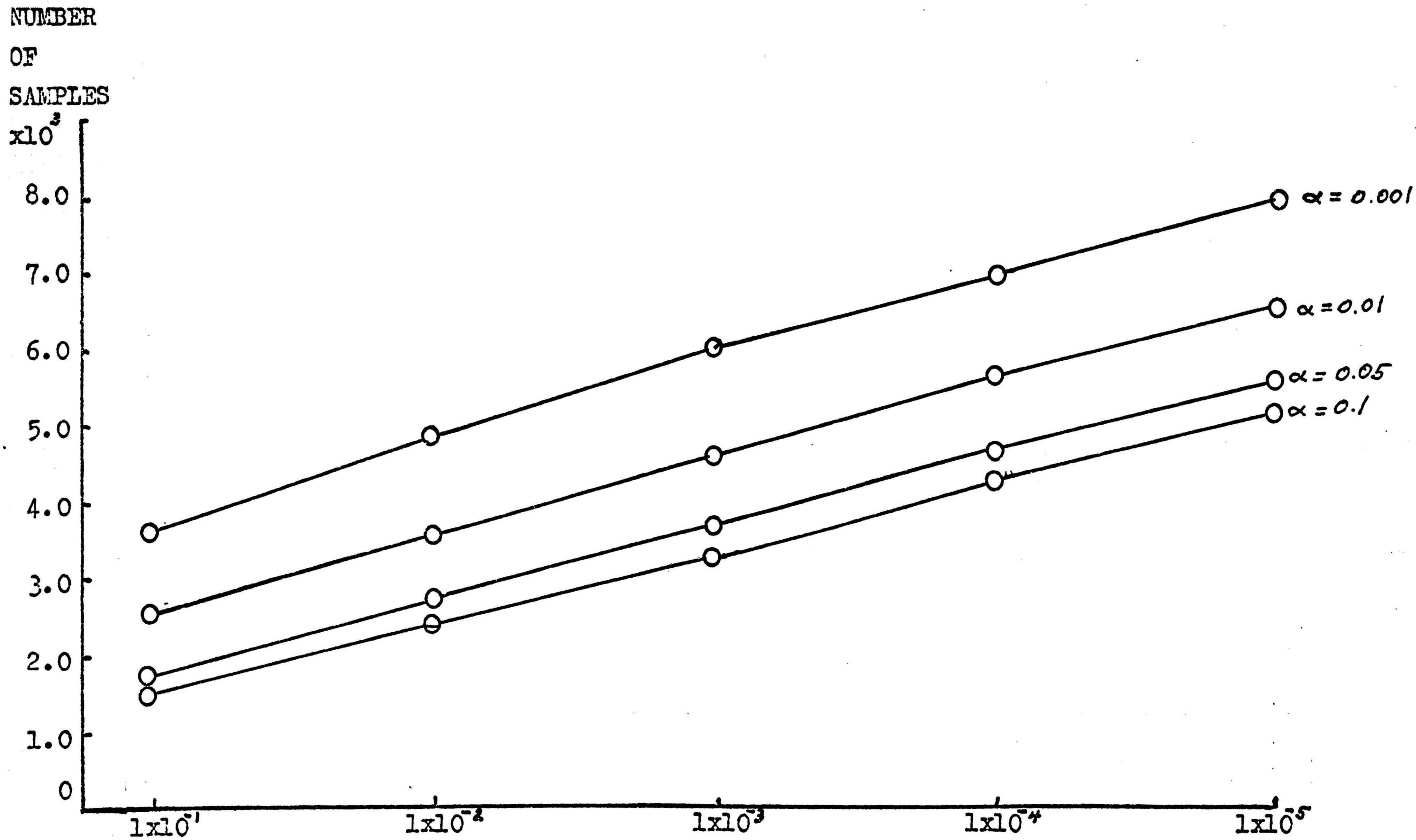

Figure 4.2. Number of Samples for C.V.M. Detector

If an α of 0.1 and a β of 0.1 are desired, it can be seen from Fig. 4.2 that approximately 1500 samples of the input waveform are necessary for detection. An IBM 360 computer was then programed to perform the actual detection of a signal buried in noise. The computer used 2000 samples in its detection scheme. To obtain these samples a number distributed according to the Rayleigh distribution was generated and used for the amplitude of the cosine wave. Another number was generated between 0 and 2π according to the uniform distribution and use as the initial phase. The computer then took 2000 samples of this cosine wave as the variable t was increased, and to each sample it added a number from a gaussian sample to correspond to noise. Then with these 2000 samples the computer performed the detection problem and decided whether or not a signal was present.

The computer worked this problem a total of 100 times (See Table VI). On 50 occasions a signal was present and on 48 of these problems the computer correctly detected the presence of the signal. On 50 occasions a signal was not present and on 46 of these problems the computer correctly detected the absence of the signal. Thus the scheme above

TABLE VI

Using C.V.M. Detector to Detect a FM Signal

$\theta = 0.5$ $\alpha = \beta = 0.1$ $w_\alpha = 0.34730$				
SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	C.V.M. STATISTIC	TRIAL	DECISION	C.V.M. STATISTIC
YES	4.47997470	1	YES	0.36988425
YES	21.74942000	2	NO	0.09819722
YES	7.93386940	3	NO	0.28066468
YES	42.83830300	4	NO	0.13581139
YES	4.65339280	5	NO	0.09181911
YES	18.87498500	6	NO	0.19121897
YES	17.21217300	7	NO	0.17029768
YES	2.87429900	8	NO	0.12150872
YES	7.02421090	9	NO	0.25140649
YES	51.88076800	10	NO	0.17569906
YES	50.47184800	11	NO	0.03097695
YES	1.31326100	12	NO	0.06335920
YES	1.02405260	13	NO	0.06144968
YES	24.15678400	14	NO	0.23448658
YES	3.06901840	15	NO	0.06794977
YES	2.57657050	16	NO	0.16795242
YES	1.61978910	17	NO	0.17688328
YES	36.64161700	18	NO	0.11350596
NO	0.08686822	19	NO	0.24716812
YES	35.92565900	20	NO	0.10069418
YES	18.33183300	21	YES	0.57908845
NO	0.15802366	22	NO	0.16196597
YES	6.59157280	23	NO	0.12691319
YES	15.76469000	24	NO	0.24968868
YES	21.52908300	25	NO	0.12950432

TABLE VI

cont.

SIGNAL PRESENT			SIGNAL ABSENT	
DECISION	C.V.M. STATISTIC	TRIAL	DECISION	C.V.M. STATISTIC
YES	76.08865400	26	YES	0.58475691
YES	51.30072000	27	NO	0.11093545
YES	41.59310900	28	NO	0.26924069
YES	12.82548900	29	NO	0.23629802
YES	59.79412800	30	NO	0.07524461
YES	26.26425200	31	NO	0.09008414
YES	51.72517400	32	NO	0.09351647
YES	22.03581200	33	NO	0.20615405
YES	10.56603300	34	NO	0.22692269
YES	65.72454800	35	YES	0.44048244
YES	10.05569800	36	NO	0.06020264
YES	9.43540760	37	NO	0.10552162
YES	5.03186990	38	NO	0.28750837
YES	6.08589170	39	NO	0.16073400
YES	0.77491903	40	NO	0.13107294
YES	13.29418200	41	NO	0.09758693
YES	30.94274900	42	NO	0.07005918
YES	6.27589990	43	NO	0.11822975
YES	42.45037800	44	NO	0.13840008
YES	19.65138200	45	NO	0.08395838
YES	65.72134400	46	NO	0.05138540
YES	7.89161010	47	NO	0.30846208
YES	18.70623800	48	NO	0.09189981
YES	64.91238400	49	NO	0.12021625
YES	15.89066500	50	NO	0.24419773

gave an α of 0.08 and a β of 0.04, which is a little better than would be expected from the values specified at the beginning of the problem.

That the results were better than expected can be explained by the fact that 2000 samples were used instead of 1500 (1500 being the number of samples needed to achieve an α of 0.1 and β of 0.1).

CHAPTER V

CONCLUSIONS

In this paper the design of a detector capable of detecting a signal which changes the noise distribution in any arbitrary fashion has been developed. After this development this detector was compared to the optimum detector for detecting two arbitrary changes in the noise. In both cases the detector of this paper (the Cramér-von Mises detector) required more samples than the optimum detectors. The C.V.M. detector required approximately twice as many samples as the t-detector for detecting shifts in the mean of the noise, and it required about four times as many samples as the χ^2-detector for detecting changes in variance.

Thus while the optimum (Neyman-Pearson) detector for any special case requires fewer samples than the C.V.M. detector for signal detection in that special case, each Neyman-Pearson detector is incapable of efficient detection for cases other than the special one for which it is designed. In contrast, the C.V.M. detector can effectively detect signals in an unlimited number of special cases without extreme loss in efficiency as compared to the Neyman-Pearson (optimum) detector. It is this flexibility without significant loss of efficiency that makes the C.V.M. detector so attractive.

In using the C.V.M. detector to detect an actual signal buried in noise in a typical situation, it was found (as expected) that the number of samples necessary for a given α and β combination depended not only on α and β but also on the signal to noise ratio θ. Theoretically this detector can be used to detect signals with extremely low ($<<1$) signal to noise ratios, but in such a case the number of samples becomes rather large. This characteristic was also verified experimentally. Also the example problem demonstrated that the C.V.M. detector can effectively detect the presence or absence of a signal in a background of noise with a prespecified probability of error.

APPENDIX A

SHOW $\sqrt{N}\,[\omega^2 - C(G)]/\sigma(G)$ HAS A NORMAL DISTRIBUTION

In order to show that $\sqrt{N}\,[\omega^2 - C(G)]/\sigma(G)$ has a normal distribution, it is necessary to start with Eq.(3.3),

$$\omega^2 = \int_{-\infty}^{\infty} \delta(x)dF(x) + 2\int_{-\infty}^{\infty} \delta(x)G(x)dF(x) - 2\int_{-\infty}^{\infty} \delta(x)G_N(x)dF(x) + \int_{-\infty}^{\infty} [G(x) - G_N(x)]^2 dF(x) .$$

It is known from Kolmogrov's Theorem that

$$\lim_{N \to \infty} \left[\Pr \left\{ \sqrt{N} \sup_{0<F(x)<1} |G(x) - G_N(x)| \geq Z \right\} \right] = 2\sum_{\nu=1}^{\infty} (-1)^{\nu-1} e^{-2\nu^2 z^2} .$$

Now it can easily be seen from above that

$$\sqrt{N} \int_{-\infty}^{\infty} [G(x) - G_N(x)]^2 dF(x) \tag{A.1}$$

tends to zero in probability.

Now looking at the third term on the right in Eq.(3.3), it follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x)G_N(x)dF(x) &= \int_{X_1}^{X_2} \tfrac{1}{N}\delta(x)dF(x) + \int_{X_2}^{X_3} \tfrac{2}{N}\delta(x)dF(x) \\ &\quad + \cdot\cdot\cdot\cdot + \int_{X_{N-1}}^{X_N} \tfrac{N-1}{N}\delta(x)dF(x) \\ &= \tfrac{1}{N}[D(X_2) - D(X_1)] + \tfrac{2}{N}[D(X_3) - D(X_2)] \\ &\quad + \cdot\cdot\cdot\cdot + \tfrac{N-1}{N}[D(X_N) - D(X_{N-1})] \\ &= (-1/N)[D(X_2) + D(X_3) + \ldots + (N-1)D(X_N)\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(x) G_N(x) dF(x) = D(\infty) - (1/N) \sum_{i=1}^{N} D(X_i) . \quad (A.2)$$

Since by definition

$$D(x) = \int_{-\infty}^{x} [F(x) - G(x)] \, dF(x) ,$$

then

$$D(x) \leq \int_{-\infty}^{x} F(x) dF(x)$$

and

$$D(x) \leq 1/2 .$$

From above and from the fact that $-\infty < x < \infty$, it can be stated that $E[D(x)]^2 < \infty$, so that by the central limit theorem

$$\sqrt{N} \left\{ (1/N) \sum_{i=1}^{N} [D(X_i) - E[D(x)]] \right\}$$

is asympototically normal with mean zero and variance given by

$$\sigma^2 [D(x)] = E[D(x)]^2 - \{E[D(x)]\}^2 ,$$

so that now

$$\sqrt{N} [\omega^2 - C(G)] = 2\sqrt{N} \left\{ (1/N) \sum_{i=1}^{N} [D(X_i) - E[D(x)]] \right\} + \sqrt{N} \int_{-\infty}^{\infty} [G(x) - G_N(x)]^2 dF(x) .$$

$\sqrt{N}[\omega^2 - C(G)]$ is the sum of an assymptotically normal random variable and one which tends to zero in probability. So it is also asympotically normal with mean zero and variance given by $4\sigma^2[D(x)]$.

APPENDIX B

VALUE OF θ WHEN a(t) HAS A RAYLEIGH DISTRIBUTION

The r.m.s. value of $a(t)\cos[\omega_j t + \phi(t)]$ must first be found before θ can be calculated. Let

$$w = a(t)\cos[\omega_j t + \phi(t)] \ .$$

Using the substitution for the cosine of the sum of two angles, the above becomes

$$w = a(t)\ \cos(\omega_j t)\cos\phi(t) - \sin(\omega_j t)\sin\phi(t)$$

$$w = a(t)[x - y]$$

where

$$x = \cos(\omega_j t)\cos\phi(t)$$

and

$$y = \sin(\omega_j t)\sin\phi(t) \ .$$

Since a(t), x, and y are independent of each other

$$\text{r.m.s.}^2(w) = \text{r.m.s.}^2(a)[\text{r.m.s.}^2(x) + \text{r.m.s.}^2(y)]. \quad \text{(B.1)}$$

The Rayleigh distribution is given by

$$a = a\ \exp(-a^2/2\sigma_s^2)\ /\sigma_s^2 \qquad 0 \leq a < \infty$$

where a is the random variable and σ_s is a parameter of the distribution. For any random variable the second moment (r.m.s.^{2}(a)) is given by

$$\text{r.m.s.}^2(a) = \int_{-\infty}^{\infty} a^2 p(a)da$$

or for this case

$$\text{r.m.s.}^2(a) = \int_{0}^{\infty} (a^3/\sigma_s^2)\ \exp(-a^2/2\sigma_s^2)da$$

or

$$\text{r.m.s.}^2(a) = 2\sigma_s^2 .$$

Now examining the second term in Eq.(B.1) yields,

$$\text{r.m.s.}^2(x) = \text{r.m.s.}^2(\cos\omega_j t)\text{r.m.s.}^2(\cos\phi(t)) \quad \text{(B.2)}$$

where

$$\text{r.m.s.}^2(\cos\omega_j t) = 1/2 \quad .$$

The uniform distribution of $\phi(t)$ is given by

$$\phi(t) = 1/2\pi \qquad\qquad 0 \leq \phi < 2$$

Then

$$\text{r.m.s.}^2[\cos\phi(t)] = 1/2\pi \int_0^{2\pi} \cos^2[\phi(t)]\,d\phi$$

or

$$\text{r.m.s.}^2[\cos\phi(t)] = 1/2 .$$

Thus from Eq.(B.2)

$$\text{r.m.s.}^2(x) = 1/4 \,.$$

Similiarily the value for r.m.s.$^2(y)$ is found to be

$$\text{r.m.s.}^2(y) = 1/4 \,.$$

Thus from Eq.(B.1)

$$\text{r.m.s.}^2(w) = \sigma_s^2$$

or

$$\text{r.m.s.}(w) = \sigma_s$$

Since θ is defined as the r.m.s. value of the signal divided by the r.m.s. value of the noise, then θ can be expressed as

$$\theta = \sigma_s/\sigma_o \tag{B.3}$$

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